Duration : 180 minutes

Linear Algebra Exam

Common part

Fall 2020

Questions

For the **multiple choice** questions, we give

$+3$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 if your answer is incorrect.

For the **true/false** questions, we give

$+1$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 points if your answer is incorrect.

The notation and terminology of this exam are those used in the exercise sheets and the lectures of the course Linear Algebra given during the Fall semester 2020.

Notation (all standard)

– $\mathbb{R}$ denotes the set of real numbers.

– For a matrix A, $a_{ij} \in \mathbb{R}$ denotes the entry of A in row i and column j.

– For a vector $\mathbf{x}$, x_i denotes the ith coordinate of $\mathbf{x}$.

– I_m denotes the $m \times m$ identity matrix.

– $\mathbb{P}_n$ is the vector space of polynomials of degree less than or equal to n.

– The scalar or inner product of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined as $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}$.

First part: Multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct response.

Question 1 : Let A be a symmetric matrix such that

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

are three eigenvectors of A associated with, respectively, the three eigenvalues 1, 0, and 2. Then

☐ $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$

☐ $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

☐ $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

☐ $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Question 2 : Given that $\lambda = 1$ is an eigenvalue of

$$\begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & -1 & 0 \\ -3 & 0 & 0 & -1 \end{pmatrix},$$

then its geometrical multiplicity (i.e. the dimension of the associated eigenspace) is

☐ 0 ☐ 1 ☐ 2 ☐ 3

Question 3 : The inverse $B = A^{-1}$ of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

is such that

☐ $b_{32} = -2/3$ ☐ $b_{23} = -2/3$ ☐ $b_{31} = -2/3$ ☐ $b_{13} = -2/3$

Question 4 : The characteristic polynomial of the matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ 4 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

is

☐ $-\lambda^3 + 5\lambda^2 - 4\lambda + 3$

☐ $\lambda^3 + 5\lambda^2 + 5\lambda - 3$

☐ $-\lambda^3 + 5\lambda^2 - 5\lambda - 3$

☐ $-\lambda^3 + 7\lambda^2 - \lambda$

Question 5 : For the LU decomposition of

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 4 \\ 1 & 0 & 3 & 2 \end{pmatrix}$$

(computed using **only** the elementary row operations of adding a multiple of a line to a line **below** itself), the resulting matrix L is such that

☐ $\ell_{42} = 1$ ☐ $\ell_{42} = 2$ ☐ $\ell_{42} = 1/3$ ☐ $\ell_{42} = -1/3$

Question 6 : Let α be a real parameter and A be the matrix

$$A = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 3 & -4 \\ 3 & 4 & \alpha \end{pmatrix}.$$

For what value of α is the rank of A less than 3?

☐ $\alpha = 3$ ☐ $\alpha = -3$ ☐ $\alpha = 0$ ☐ $\alpha = -7$

Question 7 : Consider $\mathbb{R}^3$ with the usual scalar product. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the basis of $\mathbb{R}^3$ with the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

Then applying the Gram-Schmidt algorithm to $\mathcal{B}$ yields an orthogonal basis given by the vectors

☐ $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

☐ $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -4/3 \\ 2/3 \\ -2/3 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ -1//\sqrt{3} \end{pmatrix}$

Question 8 : For the same vectors $\mathbf{v}_1$, $\mathbf{v}_2$ as in the previous question, the orthogonal projection of the vector,

$$\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

on the sub-space generated by $\mathbf{v}_1$, and $\mathbf{v}_2$ is the vector:

☐ $\begin{pmatrix} 7/3 \\ -7/6 \\ 7/6 \end{pmatrix}$ ☐ $\begin{pmatrix} 5/3 \\ 19/6 \\ -1/6 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ ☐ $\begin{pmatrix} 5 \\ 8 \\ -2 \end{pmatrix}$

Question 9 : For

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 1 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ and } \widehat{\mathbf{x}} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix},$$

If $\widehat{\mathbf{x}}$ is the least squares solution of the system $A\mathbf{x} = \mathbf{b}$, then

☐ $\hat{x}_2 = 8/7$ ☐ $\hat{x}_2 = -2/7$ ☐ $\hat{x}_2 = 3$ ☐ $\hat{x}_2 = 6/7$

Question 10 : For

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

The solution $\mathbf{x} \in \mathbb{R}^3$ of $A\mathbf{x} = \mathbf{b}$ has second component

☐ $x_2 = -3$ ☐ $x_2 = 10$ ☐ $x_2 = 6$ ☐ $x_2 = 1$

Question 11 : Consider the following bases for respectively $\mathbb{R}^2$ and $\mathbb{P}_2$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \text{ and } \mathcal{D} = \{1, t+t^2, t-t^2\}.$$

Let $T : \mathbb{R}^2 \to \mathbb{P}_2$ be the linear transformation $T(\mathbf{x}) = x_1 t + x_2 t^2$. Then the matrix of T with respect to the bases $\mathcal{B}, \mathcal{D}$ is

- ☐ $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- ☐ $\begin{pmatrix} 0 & 0 \\ 1 & 1/2 \\ 0 & -1/2 \end{pmatrix}$
- ☐ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$
- ☐ $\begin{pmatrix} 0 & 0 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$

Question 12 :
The product $C = AB$ of the matrices

$$A = \begin{pmatrix} 1 & 0 & 1 & 4 \\ 2 & 3 & 1 & 8 \\ 1 & 0 & 3 & 5 \\ 4 & 1 & -3 & -2 \\ -2 & 1 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & -2 & 4 & 5 \\ 1 & 0 & 7 & -2 & 7 \\ 3 & 1 & 1 & 3 & 0 \\ 4 & -5 & 9 & 1 & 1 \end{pmatrix},$$

is such that

- ☐ $c_{53} = 10$
- ☐ $c_{53} = -8$
- ☐ $c_{53} = 18$
- ☐ $c_{53} = 38$

Question 13 : The straight line that best approximates (in the sense of least squares) the (x, y) point data $(1, 2), (-1, 5), (0, 3)$ is

- ☐ $y = \frac{10}{3} - \frac{3}{2}x$
- ☐ $y = \frac{3}{2} + \frac{10}{3}x$
- ☐ $y = -\frac{10}{3} - \frac{3}{2}x$
- ☐ $y = \frac{10}{3} - \frac{14}{9}x$

Question 14 :
Let $\mathcal{B}$ be the basis of $\mathbb{P}_2$ given by $\mathcal{B} = \left\{-1+t, 2t-t^2, 2-t+3t^2\right\}$ and let $p \in \mathbb{P}_2$ be $p(t) = t - 8t^2$. Then the third coordinate of p relative to the basis $\mathcal{B}$ is

- ☐ $-15/7$
- ☐ $11/7$
- ☐ 15
- ☐ $-11/7$

Question 15 : Let

$$A = \begin{pmatrix} 1 & 4 & 9 & -2 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 2 & 1 & -1 & 0 \end{pmatrix}.$$

Then

☐ $\det(A) = -8$ ☐ $\det(A) = -3$ ☐ $\det(A) = 7$ ☐ $\det(A) = 0$

Question 16 : Let

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 4 \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} h \\ 2 \\ 2h \\ -2 \end{pmatrix}.$$

For what value of $h \in \mathbb{R}$ is $\mathbf{b}$ in $\operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$?

☐ $h = 1/8$ ☐ $h = -2$ ☐ $h = 1$ ☐ $h = 1/4$

Question 17 : Let $\mathcal{B} = \{1 + 3t, 2 + t^2, 4 + t + 3t^2\}$ and $\mathcal{C} = \{1, t, t^2\}$ be two bases of $\mathbb{P}_2$. If P denotes the change of coordinate matrix such that $[q]_{\mathcal{C}} = P[q]_{\mathcal{B}}$ for all $q \in \mathbb{P}_2$, then

☐ $p_{12} = 4$ ☐ $p_{12} = 3$ ☐ $p_{12} = 0$ ☐ $p_{12} = 2$

Question 18 : For the reduced echelon form R of the matrix

$$\begin{pmatrix} 2 & 1 & 3 & -4 \\ 1 & 2 & 3 & -1 \\ 3 & 2 & 1 & -2 \end{pmatrix}.$$

☐ $r_{24} = -11/12$ ☐ $r_{24} = 19/12$ ☐ $r_{24} = -19/12$ ☐ $r_{24} = 17/12$

Question 19 : The matrix

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

☐ is invertible and its inverse is orthogonally diagonalisable

☐ is invertible and its inverse is not orthogonally diagonalisable

☐ is not orthogonally diagonalisable

☐ is not invertible

Question 20 : Let k and ℓ be real parameters and

$$A = \begin{pmatrix} 1 & k & 0 \\ 5 & 2 & 1 \\ 0 & \ell & 1 \end{pmatrix}.$$

Then 6 is an eigenvalue of A when

- ☐ $5k + \ell = 20$
- ☐ $5k + \ell = 2$
- ☐ $k + \ell = 4$
- ☐ $5k + \ell = 56$

Second part: true/false questions

For each question, mark TRUE if the statement is **always true** and FALSE if it is **not always true** (i.e., it is sometimes false).

Question 21 : If each row of A is orthogonal to all the vectors in its nullspace $\mathrm{Nul}(A)$, then the matrix is symmetric.

☐ TRUE ☐ FALSE

Question 22 : The set

$$\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}) \text{ such that } a, b, c \in \mathbb{R} \right\}$$

is a subspace of the vector space $\mathcal{M}_{2\times 2}(\mathbb{R})$ of 2×2 matrices with real coefficients.

☐ TRUE ☐ FALSE

Question 23 : The vector $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ is in the range of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$, $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{pmatrix} 4 & 1 & 2 & 1 \\ 1 & -3 & 1 & 2 \\ 0 & 2 & 0 & -3 \end{pmatrix}.$$

☐ TRUE ☐ FALSE

Question 24 : For

$$A = \begin{pmatrix} 5 & 1 & -2 \\ 3 & -2 & 1 \\ 2 & 3 & -3 \end{pmatrix},$$

the vector $\begin{pmatrix} 3 \\ 11 \\ -1 \end{pmatrix}$ is in $\mathrm{Nul}(A)$ (i.e. the null space of A).

☐ TRUE ☐ FALSE

Question 25 : Given the same matrix A as in the previous question, the vector $\begin{pmatrix} 6 \\ 1 \\ 5 \end{pmatrix}$ is in $\mathrm{Col}(A)$ (i.e. the column space of A).

☐ TRUE ☐ FALSE

Question 26 : The linear system

$$\begin{cases} 2x & & - & 4z & + & 2t & = & -10 \\ & y & + & z & & & = & 2 \\ 3x & + 5y & + & 8z & - & t & = & -6 \\ 2x & + y & - & 3z & + & 2t & = & 1 \end{cases}$$

has at least one solution.

☐ TRUE ☐ FALSE

Question 27 : If the matrix A is invertible, then $A^T A$ is invertible

☐ TRUE ☐ FALSE

Question 28 : If A is a 3×3 matrix with eigenvalues -1, 1, 2, then the determinant of $A^T A$ is 2.

☐ TRUE ☐ FALSE

Question 29 :
If, as in the previous question, A is a 3×3 matrix with eigenvalues -1, 1, 2, then the matrix A is invertible, and the determinant of A^{-1} is $-1/2$.

☐ TRUE ☐ FALSE

Question 30 : Let A be a symmetric matrix with eigenpairs $(\mathbf{v}_1, \alpha), (\mathbf{v}_2, \alpha), (\mathbf{v}_3, \beta)$ i.e. eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^n$ and associated eigenvalues α, α, β, where $\alpha, \beta \in \mathbb{R}$ and $\alpha \neq \beta$. Then for any $\mathbf{w} \in \mathrm{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$, $\mathbf{w} \cdot \mathbf{v}_3 = 0$.

☐ TRUE ☐ FALSE

Question 31 : Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ be three non-zero vectors. If $\mathbf{u}$ is orthogonal to $\mathbf{v}$, and $\mathbf{v}$ is orthogonal to $\mathbf{w}$, then $\mathbf{u}$ is orthogonal to $\mathbf{w}$.

☐ TRUE ☐ FALSE

Question 32 :
If A is an 8×5 matrix such that dim Nul$(A) = 2$. Then the dimension of its row space Row(A) is 3.

☐ TRUE ☐ FALSE

Question 33 : Let A be a $m \times (m+1)$ matrix such that its column space $\mathrm{Col}\, A = \mathbb{R}^m$. Let $\mathbf{x}, \mathbf{y}$ be in $\mathbb{R}^{m+1}$ such that $\mathbf{x} \neq \mathbf{y}$ and $A\mathbf{x} = A\mathbf{y}$, and let $\mathbf{z} = \mathbf{x} - \mathbf{y}$. Then $\{\mathbf{z}\}$ is a basis of the nullspace Nul(A).

☐ TRUE ☐ FALSE

Question 34 : The following vectors in $\mathbb{R}^3$

$$\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

are linearly independent.

☐ TRUE ☐ FALSE

Question 35 :
Let A be a $m \times n$ and B be a $n \times m$ matrix such that BA is invertible. Then the rank of A is equal to n.

☐ TRUE ☐ FALSE

Question 36 : Given that the matrix

$$B = \begin{pmatrix} 1 & 0 & 0 & 47/11 \\ 0 & 1 & 0 & -49/22 \\ 0 & 0 & 1 & 3/11 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is the reduced echelon form for the matrix

$$A = \begin{pmatrix} 1 & 2 & -3 & -1 \\ 2 & 4 & 5 & 1 \\ 3 & 6 & 2 & 0 \\ -1 & 0 & 1 & -4 \end{pmatrix}.$$

then the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 47/11 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -49/22 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3/11 \end{pmatrix}.$$

form a basis for Row(A) (i.e. the rowspace of A)

☐ TRUE ☐ FALSE

Question 37 : The matrix

$$\begin{pmatrix} 3 & 2 & 0 \\ 0 & 4 & -12 \\ 0 & 0 & 5 \end{pmatrix}$$

is diagonalisable.

☐ TRUE ☐ FALSE

Question 38 : Let S be a subset of the vector space V. If $\dim V = n$ and if S spans V, then S is a basis for V

☐ TRUE ☐ FALSE

Question 39 : Let V be a vector space of dimension 2, W be a vector space of dimension 5, and $T\colon V \to W$ be an injective linear transformation. Then the dimension of the range of T is 2.

☐ TRUE ☐ FALSE

Question 40 : If A is a 6×4 matrix, then the smallest possible dimension of its nullspace $\text{Nul}(A)$ is 2

☐ TRUE ☐ FALSE